On the number of Knight's Tours

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Consider the classical problem of the Knight's tour: *find an initial square in an 8×8 chess board and a sequence of 63 movements for a knight such that the knight visits every square exactly once.* If we add the restriction that from the last square in the tour the knight should be able to reach the starting square, we have a *re-entrant solution*. In this note we prove that the total number of solutions of this problem, denoted by s, is divisible by 8 and give a rough upper bound of this quantity.

Also of interest is the number of re-entrant solutions. Each re-entrant solution is naturally associated with other 63 re-entrant solutions, obtained begining in the second, third, $\dots 63^{th}$ square of the given solution, making the corresponding movements in order to have the same sequence of squares. Noting that each of the 64 re-entrant solutions obtained gives a different re-entrant solution inverting the order, we obtain 128 re-entrant solution.

In terms of graph theory ([1]), if we say that (i) a *path* is a sequence of different squares visited by a knight, (ii) a path is *closed* when from the last square of the sequence the knight can go to the first one, (iii) a path is *Hamiltonian* when it includes all the squares; and observe that we are counting *directed* paths, in view that we consider as different solutions to our the problem the same sequence of squares visited in the direct and inverse order, it can be said that s is the number of *directed Hamiltonian paths*.

The Knight's tour problem has a very long history. Before computers, finding solutions and describing its properties was a challenge attracting the attention of many relevant mathematicians, including Euler, Legendre and Vandermonde [2], [3]. In present times, this well known problem is chosen in order to test computational tools for counting problems with the help of computers. As an example, I. Wegener (in [4]) and Mc Kay (see comments to [5]) obtained independently that the number of classes of re-entrant solutions (not taking into account the order and the initial square) is $w = 13.267.364.410.532$.

The divisibility property that we present is proved elementary by considering symmetries and rotations of the chessboard. Denote by $s(ij)$ be the number of tours begining in the square (i,j) $(i,j = 1,2,\dots,8)$. It is clear that $s = \sum_{i,j=1}^{8} s(ij)$. Based on invariance under rotations of the chessboard we obtain that

$$s = 4 \sum_{i,j=1}^{4} s(ij).$$

Symmetry with respect to the diagonal from $(1,1)$ to $(8,8)$, gives that $s(ij) = s(ji)$ $(i,j = 1,2,3,4)$, and we conclude that

$$s = 4\{s(11)+s(22)+s(33)+s(44)+2[s(12)+s(13)+s(14)+s(23)+s(24)+s(34)]\}. \tag{1}$$

In this way, the computation of the total number of tours is reduced to the computation of tours begining in the squares indicated in Figure 1. It remains

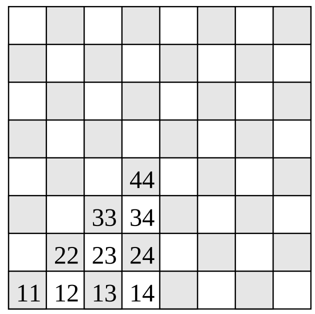

Figure 1: Relevant initial squares

to see that $s(11), s(22), s(33)$ and $s(44)$ are even numbers. Denote $s(ij,kl)$ the number of tours begining in (i,j) with first visit to (k,l). The same symmetry argument gives

$$\begin{aligned}
s(11) &= 2s(11,23), \\
s(22) &= 2[s(22,14) + s(22,34)] \\
s(33) &= 2[s(33,12) + s(33,14) + s(33,25) + s(33,45)], \\
s(44) &= 2[s(44,23) + s(44,25) + s(44,36) + s(44,56)].
\end{aligned}$$

In conclusion

$$s = 8[s(11,23) + s(22,14) + s(22,34) + s(33,12) + s(33,14)$$

$$+s(33,25) + s(33,45) + s(44,23) + s(44,25) + s(44,36) + s(44,56)$$

$$+s(12) + s(13) + s(14) + s(23) + s(24) + s(34)],$$

concluding the proof.

Let us add some comments on upper bounds for s. The computation of w gives that $128w \sim 1.68 \times 10^{15}$ is a lower bound for s.

As the number of possible movements for the knight is 168, and a (non-directed) solution is a set of 63 different movements, we obtain that $2 \times \binom{168}{63}$ is an upper bound for s (see [2]). This is approximately 2.35×10^{47}. Another rough upper bound can be obtained as the product of the 64 initial squares by the number of possible movements from each square within a tour, giving $64.2^4 3^8 4^{20} 6^{16} 8^{16}$. This gives approximately 5.86×10^{45}. If we take into account the partition of the set of solutions introduced above, and the following facts: (a) excepting in the first case, the number of possible movements from a square is the total of accessible squares (2,3,4,6 or 8) minus one. (b) each corner (i.e. $(1,8)$) is connected with two squares with 6 possible movements (in this case $(2,6)$ and $(3,7)$) but, as the solution must visit the corner only one 5 connected to each corner must be counted (excepting solutions begining in $(1,1)$). (c) For the last three movements there are at most two possibilities. This means that the last four factors in the product can be replaced by 1, or by 2 when the corresponding four squares form a closed path. The case with minimum product (taking into account (b)) is 3-2-1-1 in a corner. All other cases gives bigger product, and the closed paths give products bigger than 12. Denote then

$$k = 1^{10}2^7 3^{19} 5^{12} 7^{16}.$$

Taking into account the initial squares, and noting that for $s(11)$ the 1 in $(3,2)$

1	**1**	3	3	3	3	2	1
2	3	**1**	5	5	5	3	2
1	5	7	7	7	7	1	3
3	5	7	7	7	7	5	3
3	5	7	7	7	7	5	3
3	1	7	7	7	7	5	3
2	3	5	5	5	1	3	2
1	2	3	3	3	3	2	1

Figure 2: Numbers of accessible squares within the tour. 1 and 5 can be interchanged in squares connected with each corner. In bold are indicated four connected squares giving the minimum possible product (then replaced by 1).

must be replaced by 4 possible movements we obtain the following bounds:

$$\begin{array}{ll} s(11) \leq 8k & s(22) \leq 4k/3 \\ s(33) \leq 8k/7 & s(44) \leq 8k/7 \\ s(12) \leq 3k/2 & s(13) \leq 4k/3 \\ s(14) \leq 4k/3 & s(23) \leq 6k/5 \\ s(24) \leq 6k/5 & s(34) \leq 8k/7 \end{array}$$

Summing up in (1), we obtain $s \leq \frac{11356}{105}k$. This is approximately 1.305×10^{35}.

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