

# Counting Knight's Tours through the Randomized Warnsdorff Rule

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## Abstract

We give an estimate of the number of geometrically distinct open tours  $\mathbf{G}$  for a knight on a chessboard. We use a randomization of Warnsdorff rule to implement importance sampling in a backtracking scheme, correcting the observed bias of the original rule, according to the proposed principle that “most solutions follow Warnsdorff rule most of the time”. After some experiments in order to test this principle, and to calibrate a parameter, interpreted as a distance of a general solution from a Warnsdorff solution, we conjecture that  $\mathbf{G} = 1.22 \times 10^{15}$ .

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# 1 Introduction

In this paper we give an estimate of the number of open Knight Tours in a chessboard, based on the heuristics given by the classical Warnsdorff rule, used to implement importance sampling in a backtracking algorithm, following [Knuth 1975]. The historical development of the Knight Tour problem is beyond the possibilities of the present work, and the interested reader is suggested to visit George Jelliss’s web page and the references therein [Jelliss 2003], from where we borrow our notation. Let us define precisely the quantities we are interested in:

- **G** is the number of **G**eometrically distinct open tours.
- **T** is the number of open **T**our diagrams, by rotation and symmetry  $\mathbf{T} = 8 \mathbf{G}$ .
- **N** is the number of open tour **N**umberings:  $\mathbf{N} = 2 \mathbf{T} = 16 \mathbf{G}$  since each can be numbered from either end.
- **D** is the number of closed tour **D**iagrams.

These parameters refer to the standard  $8 \times 8$  chessboard. When other sizes of (squared) chessboards are considered, we include a subscript in our notation, for instance  $\mathbf{N}_6$  is the number of open tour numberings in a  $6 \times 6$  chessboard.

In the terms just introduced our purpose is to give an estimate of **G**. A relevant exact computation related to this problem was carried out independently in [McKay 1997] and in [Wegener 2000], who computed the number  $\mathbf{D} = 13, 267, 364, 410, 532$ .

# 2 Importance sampling through Warnsdorff rule

Warnsdorff rule was proposed in 1823 in order to find open tours in a chessboard. It states: “at each step, choose the square with the lowest number of possible continuations; if a tie occurs it may be broken arbitrarily” [Warnsdorff 1823]. This heuristic rule is incredibly effective to find solutions. Only 150 years after being formulated, with the help of a computer, it was reported that the rule can fail, in respect to its second part [Guik 1983]. Modern numerical experiments showed that this rule is *biased*, in the sense that it finds solutions in an extremely small subset of the set of solutions, fact that we confirm.

Our departure point in order to estimate the total amount of open tours, is the principle that “most solutions follow Warnsdorff rule most of the time”, i.e. a typical solution is a concatenation of short paths satisfying the rule within them, that, when concatenated violate the rule. According to this principle we decompose the number **G** as the sum  $\mathbf{G} = \mathbf{G}_0 + \dots + \mathbf{G}_{62}$  where we denote by  $\mathbf{G}_k$  ( $k = 0, 1, \dots, 62$ ) the number of geometrically distinct knight tours that violates the rule exactly  $k$  times. In particular  $\mathbf{G}_0$  is the number of knight tours that follows Warnsdorff rule (including any of the possibilities in case of tie). In section 4 we provide some empirical evidence supporting this principle, giving an estimation of  $\mathbf{G}_k$  for different values of  $k$ . We obtain the largest number when  $k = 13$ , and significant numbers in a range from  $k = 5$  to  $k = 20$ .

In consequence, following [Knuth 1975] and these considerations, we implement a randomized backtracking algorithm, that we call the “Randomized Warnsdorff Rule”. We begin by

			44				
		33	34				
	22	23	24				
11	12	13	14				

Figure 1: Relevant initial squares

fixing the ten geometrically different initial squares (see Figure 2) and, at step  $n$ , we assign probabilities  $p(n, \alpha, i(n), j)$  to a jump from the current square  $i(n)$  to each connected square  $j$ , proportional to  $N_j^{-\alpha}$  where  $N_j$  is the number of free accessible squares from  $j$  and  $\alpha$  is an *Importance Sampling* (IS) parameter. The estimation of the number of tours for each run is

$$\begin{cases} \prod_{n=1}^{63} \left( p(n, \alpha, i(n), i(n+1))^{-1} \right) & \text{when the algorithm finds the solution } \{i(n)\}, \\ 0 & \text{when the algorithm does not complete a tour.} \end{cases}$$

This is an *unbiased* estimate of the number of tours beginning at  $i(1)$ , for all positive probability assignments [Knuth 1975]. The problem is then to choose the probabilities  $p(n, \alpha, i, j)$  in order to achieve the best possible *variance reduction*, and here we use Warnsdorff heuristics.

Two important cases of our scheme are  $\alpha = 0$ , corresponding to plain Monte Carlo simulation in the graph, and  $\alpha = \infty$ , that gives maximum (equal) probability to the nodes with minor amount of free nodes, and null probability to the others, i.e. the original Warnsdorff rule with equiprobability in case of tie, giving a biased estimate.

The first step of our work consists in calibrating the parameter  $\alpha$ , that measures the distance from a generic solution from a Warnsdorff solution, running our randomized Warnsdorff algorithm to estimate three known quantities,  $\mathbf{N}_5$ ,  $\mathbf{D}_6$  and  $\mathbf{D}$ . It is important to note (and it has been noticed in the literature) that the optimal  $\alpha$  is sensible to the size of the chessboard. We report the results in Section 3

The second step, once we calibrate  $\alpha = 1.5$ , consists in running a relatively long sampling experiment to obtain our final estimate of  $\mathbf{G}$ , presented in Section 5. We use this run also to compute the results presented in Section 4.

### 3 Parameter tuning

In this section we present the results of some numerical experiments that were designed in order to validate the proposed method and at the same time to tune  $\alpha$ , the importance sampling parameter.

We estimate three different known quantities using the randomized Warnsdorff rule:  $\mathbf{N}_5 = 1728$ , obtained in [Planck 1908] and [Planck 1909],  $\mathbf{D}_6 = 98626$  obtained in [Duby 1964], and the McKay-Wegener number  $\mathbf{D}$ .

To define the experimental design, it is necessary to fix a certain number of parameters, in particular the sampling size for the Monte Carlo experiments, the number of replications with different seeds, and the importance sampling parameter  $\alpha$ . The two first parameters were fixed taking into account some preliminary experiments and the need to complete work in a reasonable computing time; the sampling size was then fixed to  $5 \times 10^5$  for the  $5 \times 5$  board and to  $10^6$  for the other two cases, and 21 replications were made with different seeds. Even if  $\alpha$  is a continuous parameter, it is necessary to decide on a (small) set of values for employing in the experiments. It is important to include the case where there is no IS at all ( $\alpha = 0$ ), as well as a relatively wide range of values where the method has different behaviors. We also included  $\alpha = -1$ , which corresponds to an IS scheme which favors tours “far away from Warnsdorff rule”. Based on the same preliminary experiments, we choose to take  $\alpha$  in the set  $\{-1, 0, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5, 8, 10\}$ . The total number of runs is then 756 (three problems, 12 different values for  $\alpha$ , 21 independent replications with different seeds for each of these). The Monte Carlo method was programmed using C (compiled with gcc 3.3.3 and 3.4.3), and all the experiments were run on an personal computer (CPU 1.70GHz, RAM 224508 kB) running Linux. Each experiment took in the order of 100 CPU seconds on the  $5 \times 5$  board, 2000 CPU seconds on the  $6 \times 6$  board and 15000 seconds on  $8 \times 8$  boards (as reported by the `times()` function of the “`sys/times.h`” library). A summary of the results obtained is shown in

$\alpha$	$\min_i \{\mathbf{N}_5(i)\}$	$\widehat{\mathbf{N}}_5$	$\max_i \{\mathbf{N}_5(i)\}$	$\sigma$	<i>time</i>
-1	314	<b>2120</b>	8561	2168	75.3
0	1578	<b>1732</b>	1887	83	56.0
0.5	1663	<b>1716</b>	1789	28	129.6
1	1693	<b>1729</b>	1758	13	100.5
1.5	1714	<b>1729</b>	1742	8.0	138.2
2	1720	<b>1729</b>	1741	6.2	118.5
2.5	1722	<b>1728</b>	1738	6.0	153.8
3	1712	<b>1728</b>	1740	6.8	118.7
4	1702	<b>1726</b>	1752	12	128.7
5	1686	<b>1732</b>	1795	27	129.0
8	1279	<b>1676</b>	2589	339	133.8
10	1002	<b>1866</b>	11310	1871	124.4

Table 1: Quality of estimations of  $\mathbf{N}_5=1728$  as function of the IS parameter  $\alpha$ .

Tables 1, 2, and 3 respectively for the three problems studied. Given a problem, for instance, the estimation of  $\mathbf{N}_5$ , the number of open tours numberings in a  $5 \times 5$  chessboard, and a value for  $\alpha$ , we have  $i = 1, \dots, 21$  independent replications, each returning an estimation numbered through  $\mathbf{N}_5(i)$ , and an estimation  $\sigma(i)$  of the standard deviation (i.e. the square root of the variance) of this  $i$ -th estimation. The final estimation, the mean in  $i$ , is denoted  $\widehat{\mathbf{N}}_5$ . Each table has five columns, the first one corresponding to the  $\alpha$  values, and the following ones showing

the minimum, the estimation, and maximum values for the 21 estimations, an estimation  $\sigma$  of the standard deviation of our estimate, and the mean of the execution times. The same analysis is performed with  $\mathbf{D}_6$  and  $\mathbf{D}$ .

It comes as no surprise to observe that the first two problems are rather easy, and a wide range of  $\alpha$  values result in estimations quite near to the exact value. It is clear that the variance of the estimation depends strongly on  $\alpha$ ; in both cases the best value is observed when  $\alpha = 2$  (but  $\alpha = 1.5$  and  $\alpha = 2.5$  come very close). High values for  $\alpha$  seem to greatly affect the precision of the method (this can be specially be seen in the last two lines for the  $6 \times 6$  board, where the estimations are quite bad). Also,  $\alpha = -1$  leads to inconsistent results.

The standard (no importance sampling) Monte Carlo results can be seen in the  $\alpha = 0$  line. In both cases the standard method gives quite consistent estimates for the number of tours, but using Importance Sampling we can attain a variance orders of magnitude smaller.

From these experimental results, it can be seen that in the three cases under study the proposed method works well and is more robust and more efficient than the standard Monte Carlo, which in some cases does not give meaningful estimations of the desired parameters. Fixing  $\alpha$  can be a problem, as the method is clearly dependent on this parameter, but fortunately it seems as if the choice of  $\alpha$  around 1.5 or 2 is rather robust, as those are the values with best results in these three problems. As we want to estimate in the bigger chessboard, we decide ourselves for  $\alpha = 1.5$ . Computing the number of closed tour diagrams on the  $8 \times 8$  board is

$\alpha$	$\min_i\{\mathbf{D}_6(i)\}$	$\widehat{\mathbf{D}}_6$	$\max_i\{\mathbf{D}_6(i)\}$	$\sigma$	<i>time</i>
-1	0	<b>0</b>	0	0	1.142E+03
0	7444	<b>10113</b>	13920	1387	9.207E+02
0.5	9324	<b>9945</b>	10547	220	2.076E+03
1	9718	<b>9861</b>	9960	64	1.649E+03
1.5	9816	<b>9861</b>	9925	31	2.312E+03
2	9808	<b>9862</b>	9887	24	2.002E+03
2.5	9814	<b>9871</b>	9906	26	2.783E+03
3	9799	<b>9866</b>	9929	37	2.162E+03
4	9690	<b>9896</b>	10521	191	2.416E+03
5	9231	<b>10074</b>	14568	1105	2.444E+03
8	5252	<b>8234</b>	17429	2610	2.761E+03
10	3162	<b>5866</b>	25270	4888	2.575E+03

Table 2: Quality of estimations of  $\mathbf{D}_6 = \mathbf{9862}$  as function of the IS parameter  $\alpha$ .

a more demanding problem, and the standard (no importance sampling) method can not find good estimates (nor can do it the method with  $\alpha = -1$ ). In general, the IS estimates quality depends heavily on  $\alpha$ , and only when  $0.5 \leq \alpha \leq 3$  we find estimation values that approach the exact value. Here the setting  $\alpha = 1.5$  leads to the lowest variance.

$\alpha$	$\min_i\{\mathbf{D}(i)\}$	$\widehat{\mathbf{D}}$	$\max_i\{\mathbf{D}(i)\}$	$\sigma$	$time$
-1	0	<b>0</b>	0	0	8.517E+03
0	0	<b>2.992E+13</b>	3.999E+13	8.82E+13	5.912E+03
0.5	1.061E+13	<b>1.386E+13</b>	1.692E+13	1.56E+12	1.339E+04
1	1.253E+13	<b>1.320E+13</b>	1.391E+13	3.48E+11	1.038E+04
1.5	1.280E+13	<b>1.318E+13</b>	1.398E+13	2.36E+11	1.507E+04
2	1.248E+13	<b>1.324E+13</b>	1.463E+13	4.17E+11	1.471E+04
2.5	1.225E+13	<b>1.315E+13</b>	1.534E+13	6.65E+11	1.839E+04
3	1.104E+13	<b>1.327E+13</b>	2.456E+13	3.12E+12	1.554E+04
4	7.107E+12	<b>1.122E+13</b>	2.413E+13	4.35E+12	1.717E+04
5	3.228E+12	<b>7.029E+12</b>	3.658E+13	7.10E+12	1.729E+04
8	7.694E+10	<b>3.472E+11</b>	1.571E+12	4.31E+11	1.763E+04
10	1.013E+10	<b>1.687E+11</b>	2.897E+12	6.28E+11	2.165E+04

Table 3: Quality of estimations of  $\mathbf{D} = \mathbf{13, 267, 364, 410, 532}$  as a function of  $\alpha$ .

## 4 On the set of open tours

In this section we present some empirical evidence supporting our departure point, that is, that “most solutions follow Warnsdorff rule most of the time”. Firstly, we present in Figure 2 a histogram where we show the estimations of  $\mathbf{G}_k$  obtained in the final simulation (described in next section), in the case and  $\alpha = 1.5$ , with  $10^9$  replications.

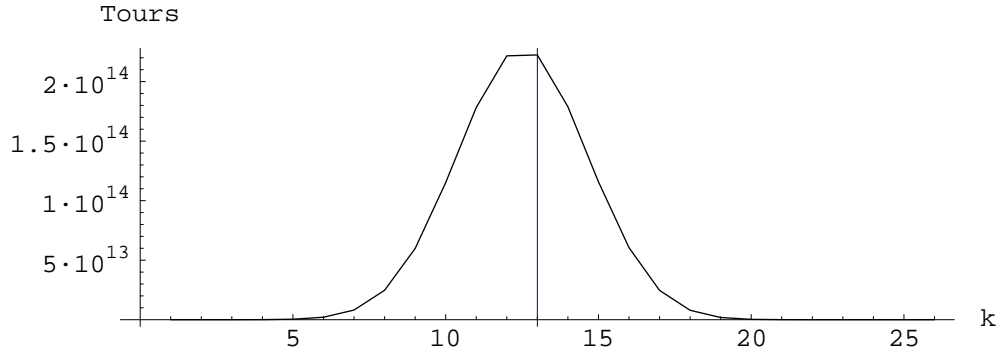


Figure 2: Estimated number of tours  $\mathbf{G}_k$  for  $k = 0, \dots, 25$  ( $N = 10^9$ ).

In the Table 4 we present the corresponding values plotted in the Figure 2. It is concluded that values from  $\mathbf{G}_5$  to  $\mathbf{G}_{20}$  concentrate the 99.9892% (practically all) solutions, and also that  $\mathbf{G}_0 \ll \mathbf{G}$ , i.e. the number of Warnsdorff solutions is several orders smaller than the number of solutions.

$k$	$\mathbf{G}_k$	$k$	$\mathbf{G}_k$	$k$	$\mathbf{G}_k$	$k$	$\mathbf{G}_k$
0	1.066E+05	6	2.084E+12	12	2.216E+14	18	7.920E+12
1	2.357E+07	7	8.110E+12	13	2.224E+14	19	1.845E+12
2	5.095E+08	8	2.475E+13	14	1.789E+14	20	3.063E+11
3	6.935E+09	9	5.980E+13	15	1.158E+14	21	6.020E+10
4	6.300E+10	10	1.152E+14	16	6.050E+13	22	1.121E+09
5	4.160E+11	11	1.783E+14	17	2.462E+13	23	0

Table 4: Estimations of  $\mathbf{G}_k$  corresponding to Figure 2

## 5 On the number $\mathbf{G}$ of open tours

In this section we present our main results, corresponding to the final numerical experiment. In this case we perform the randomized Warnsdorff algorithm (as explained in 2), with  $\alpha = 1.5$ , as suggested by the experiments exposed in section 3. We choose a sample size corresponding to  $10^9$  size, as this seems to provide a reasonable result in an acceptable computational time. In our case CPU time was  $6.5 \times 10^5$  seconds  $\sim 7.5$  days, on a similar personal computer.

In our runs we distinguish according to the 10 different initial squares (see Figure 2), and, in the four diagonal squares we estimate half of the solutions, based on the symmetry. The dynamic nature of Warnsdorff rule does not seem to give a simple way of exploiting the “go and return” symmetry, and, in consequence, our crude algorithm gives an estimation of  $2\mathbf{G}$ . To have an idea of the error of our method we counted also the closed tours, obtaining the estimation  $\hat{\mathbf{D}}=1.326390\text{E}+13$  (compare with the exact value  $\mathbf{D}=1.3267364410532\text{E}+13$ ) with a standard deviation  $\sigma=3.03\text{E}+10$ .

Our final results are presented in Table 5. Based on these final results we construct a

$\hat{\mathbf{G}}$	$\mathbf{G}$	$\sigma$
1.222801E+15	Unknown	8.26E+11

Table 5: Estimation of  $\mathbf{G}$  (sample size  $10\text{E}+9$ ).

confidence interval for  $\mathbf{G}$ . Taking into account that for a normal random variable  $X$  with mean  $\mathbf{G}$  and variance  $\sigma$  we have  $\mathbf{P}(\mathbf{G} - 3\sigma < X < \mathbf{G} + 3\sigma) = 0.997$ , we conclude that  $\mathbf{G} \in [1.220, 1.225] \times 10^{15}$  and from there our main conclusion  $\mathbf{G} = 1.22 \times 10^{15}$ .

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